08

LOCKERS

Heuristics to highlight:

- Act it out
- Consider a simpler problem (smaller numbers)
- Look for patterns
- Restate the problem in another way
- Make a systematic list.

08

The new school has exactly 343 lockers numbered 1 to 343, and exactly 343 students. On the first day of school, the students meet outside the building and agree on the following plan. The first student will enter the school and open all the lockers.

The second student will then enter the school and close every locker with an even number. The third student will then 'reverse' every third locker; i.e. if the locker is closed, he will open it, and if the locker is open, he will close it.

The fourth student will reverse every fourth locker, and so on until all 343 students in turn have entered the building and reversed the relevant lockers. Which lockers will finally remain open?

SUITABLE HINTS FOR POLYA STAGES I, II AND IV

I) UNDERSTAND THE PROBLEM

(c) Write down the heuristics you used to understand the problem.

Attempt 1 *Act it out* – Imagine you are the first few students, and act out what they will do to the first few lockers.

Attempt 2 *Restate the problem in another way* - What type of number causes the corresponding locker to be touched an odd number of times? What feature(s) of a number causes the corresponding locker to be touched? What type of number has an odd number of factors?

II) DEVISE A PLAN

(a) Write down the key concepts that might be involved in solving the problem.
Factors of an integer.

(c) Write out each plan concisely and clearly.

Plan 1 *Consider a simpler problem (smaller numbers)*, Use a suitable representation, Look for patterns - Consider the first 10 lockers, and examine whether they will be closed by the end when all the students have entered the school; use a suitable representation to depict the opening and closing of the lockers; look for patterns to see which lockers remain open.

Plan 2 *Make a systematic list, Look for patterns* – list the factors of some numbers and try to understand why a square has an odd number of factors while the others have even numbers.

IV) CHECK AND EXPAND

(a) Write down how you checked your solution.
Verify for a few cases.
(b) Write down a sketch of any alternative solution(s) that you can think of. Use prime factorisation to prove that a natural number is a square if and only if it has an odd number of factors.
(c) Give at least one adaptation, extension or generalisation of the problem.

Adaptation 1:
The i-th student reverses every i-th locker except Locker i.

Adaptation 2:
The i-th student reverses every locker whose number is a factor of i.

Generalisation 1:
There are n lockers.

Generalisation 2:
There are n lockers. How many lockers remain open?

Extension:
There are m stages to finally open the locker (for example: put the key in the lock, turn the key, pull down the handle, etc.). The i-th student goes to every i-th locker and acts out the next stage, and if the locker is fully open, closes and locks it. Which lockers remain open in the end?

SOLUTIONS AND ASSESSMENT NOTES

SOLUTION #1

Let n be a positive integer. Let a_1, a_2, …, a_k be all the k distinct factors of n with a_1 < a_2< … < a_k. Observe the following:

i) *n* is the product of the *i*-th smallest factor a_i and the *i*-th largest factor a_{k+1-i}, i.e. $n = a_i a_{k+1-i}$, where $a_i \leq a_{k+1-i}$.

ii) If *n* is not a square, for $n = a_i a_{k+1-i}$, we have

$$a_i < \sqrt{n} \text{ and } a_{k+1-i} > \sqrt{n}$$

This implies that each factor has a ‘partner’ factor distinct from itself such that the product of the two factors is n. Hence the number of factors must be even.

iii) If *n* is a square, then √*n* is an integer. As in (ii), each factor less than √*n* has a ‘partner’ factor distinct from itself such that the product of the two factors is n.
Observe that

$$n=\sqrt{n}\times\sqrt{n} \text{ and so this factor } \sqrt{n}$$

has no distinct ‘partner’. Hence the number of factors must be odd.
Now each locker will be ‘touched’ by a student whose number is a factor of the number of the locker. A square will be touched by an odd number of students because it has an odd number of factors as shown above. In the sequence of ‘open’ followed by ‘close’, an odd number of actions will end in the locker eventually open. A non-square will by the result above be closed at the end.
Hence, the lockers that will remain open will be those whose number is a square, i.e. when there are 343 students, they are 1, 4, 9, 16, 25, 36, 49, …, 324.

SOLUTION #2

If a number has prime factorization

$$n=p_1 a^1 p_2 a^2 \ldots\ldots p_n a^n,$$

then the total number of factors (the numbers 1 and itself included) is

$$(a_1+1)(a_2+1)(a_3+1)\ldots(a_n+1).$$

If this number is odd, then all the numbers a_1, a_2,….., a_n must be even, hence n must be a perfect square.

POSSIBLE STUDENT RESPONSES

Students may make only incomplete observations, leading to wrong generalizations. For example, prime numbered lockers are closed, hence all the other numbered lockers are open.

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Assessment notes

Observing from the listing of factors that factors occur in pairs and so the only numbers that have the odd number of factors are the perfect squares is only a partially correct solution.

Any systematic word or diagrammatic explanation that the factors are divided into two groups by $\sqrt{n}$ with pairs formed from one of each set and that $\sqrt{n}$ forms a pair with itself for square n can be considered correct